Averaging Techniques and Their Application to Orbit Determination Systems

C. E. VELEZ* AND A. J. FUCHS†
NASA Goddard Space Flight Center, Greenbelt, Md.

The theory of numerical averaging and analytical averaging will be reviewed and the application of these techniques to orbit and parameter estimation problems will be presented. Comparisons will be made between utilizing mean elements vs tracking data as the observation types. Results will be presented comparing the accuracy and efficiency of the combined orbit estimation and orbit prediction problem using averaged equations of motion, the Cowell equations of motion, and the Brouwer general perturbation theory. The problem of converting the averaged element space back to osculating element space for orbit operations will also be discussed.

I. Introduction

THE efficient study of the long-term behavior of satellite motion in the presence of complex perturbations by the method of averages is well known and widely used. This concept has appeared in many forms and applications over the last decade or so, particularly in mission analysis[1–8] and geodetic parameter estimation[9–12] type software systems.

The purpose of this paper is to review the basic concepts and assumptions of averaging as applied to these problems and to look at possible further applications and improvements of the theory. For example, a problem frequently encountered in the application of averaging in orbit estimation and prediction is the determination of the appropriate initial values ("mean elements") to be used to propagate the average orbit. A technique is evaluated for the determination of such elements which uses tracking data directly and which does not require the modeling of short-period oscillations in the orbital elements. General techniques for converting given "osculating" elements to "mean" elements will also be discussed.

Section II presents the classical first-order theory of averaging. Much of this discussion appears in a set of lecture notes by W. T. Kyner[13] and in the Lorell reports.[1,5] Reference 1 also contains an excellent discussion on second-order averaging. The classical references to the method of averages are the books of Kyrlov and Bogoljubov,[18] Mitropolsky,[19] and the recent book of Nayfeh.[20]

From the following and other discussions we can see the intimate conceptual relationship between the classical general perturbation theory based on the von Zeipel approach[14,15] and the classical averaging theory, both striving to eliminate high-frequency motion from the dynamical system. Further, we can see the distinction between so-called analytical averaging, in which the averaging integrals of the dynamics are computed analytically over the orbital period, eliminating the short-period terms, and numerical averaging, in which the definite integrals are computed numerically. Advantages of both approaches will be discussed. Finally, in contrast to the classical methods of averaging which generally hold the elements or variables fixed during the averaging procedure, thereby losing low-frequency motion resulting from interaction with high-frequency terms, a modified technique is proposed which allows the modeling of such terms by use of numerical quadratures and a first-order theory.

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* Head, Systems Development and Analysis Branch.

† Head, Orbital Mechanics Section.

Following this general discussion, the systems of orbital equations frequently used with averaging will be presented and a discussion of the applications of such equations in the areas of mission analysis, geodetic mean element studies and orbit estimation will be given. Several techniques for determining initial conditions for the averaged dynamics will be presented. Finally, some numerical results of 1) a comparison of the efficiency and accuracy of several orbit propagation schemes in an orbit estimation process; 2) a comparison of "tracking data determination" of mean elements with elements obtained by a more classical approach; 3) an error analysis of the averaging interval in the presence of a rotating primary, and the averaged orbit in the presence of complex perturbations.

II. Averaging in Orbital Dynamics

The basic premise underlying averaging in this area is that a typical orbital element undergoes a long-period and secular-type motion (steady-state) upon which short-period fluctuations are superimposed, and that these low and high-frequency motions can be separated and modeled independently. The advantage of this separation is that this long-period and secular motion can be easily computed numerically and yields a uniform approximation of the true motion; i.e., the averaged element differs from the actual element by a high-frequency, low-amplitude periodic function. Hence, averaging yields an efficient technique for computing the evolution of an orbit in the presence of complex perturbations such as geopotential resonance and drag forces over many revolutions, thereby playing an important role in mission analysis, and in the study of geodetic parameters causing such long period and secular motion. More will be said of applications later.

The basic concepts can be seen by examination of an initial value problem of the form

$$\begin{aligned} \dot{x} &= \varepsilon f(x, y) \\ \dot{y} &= h(x) + \varepsilon g(x, y) \\ x(t_o) &= x_o,\ y(t_o) = y_o \end{aligned} \tag{1}$$

where the functions f, g, and h are sufficiently smooth and periodic in y of period 2π. It should be clear that this system is analogous to the orbital variation of parameter equations, x representing the slow variables (e.g., a, e, i, Ω, ω) and y the "fast" variables; e.g., a mean or true anomaly. The functions f and g may model perturbations due to nonspherical potential, drag, etc. The system (1) is generally difficult to integrate numerically due to the presence of the y variables in the right-hand sides. The first-order averaging concept requires the construction of a transformation

$$(x, y) \xrightarrow{S} (\bar{x}, \bar{y})$$

of the form

$$x = \bar{x} + \varepsilon X(\bar{x}, \bar{y}) \qquad y = \bar{y} + \varepsilon Y(\bar{x}, \bar{y}) \tag{2}$$

where X and Y are periodic in $\bar{y}$, so that system (1) becomes

$$\dot{\bar{x}} = \varepsilon F(\bar{x}) + O(\varepsilon^2) \qquad \dot{\bar{y}} = h(\bar{x}) + \varepsilon G(\bar{x}) + O(\varepsilon^2) \tag{3}$$

i.e., to first order in ε, the fast variable y has been eliminated from the differential equations. The appropriate initial values to be used with system (3) are solutions of the implicit equations

$$\bar{x}_o = x_o - \varepsilon X(\bar{x}_o, \bar{y}_o) \qquad \bar{y}_o = y_o - \varepsilon Y(\bar{x}_o, \bar{y}_o) \tag{4}$$

which can be obtained by iteration if the generating functions X and Y are known, or by other numerical techniques if they are not. Differentiating Eq. (2) and equating first-order terms, we find that X and Y satisfy the equations

$$\frac{\partial X}{\partial \bar{y}} h(\bar{x}) = f(\bar{x}, \bar{y}) - F(\bar{x}) \qquad \frac{\partial Y}{\partial \bar{y}} h(\bar{x}) = \frac{\partial h}{\partial \bar{x}}(\bar{x}) X(\bar{x}, \bar{y}) + g(\bar{x}, \bar{y}) - G(\bar{x}) \tag{5}$$

and therefore will have y-periodic solutions only if the right-hand sides have zero mean, i.e.,

$$F(\bar{x}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\bar{x}, \bar{y})\, d\bar{y} \qquad G(\bar{x}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\bar{x}, \bar{y})\, d\bar{y} \tag{6}$$

In standard first-order theories, these integrals are evaluated holding $\bar{x}$ fixed during the quadrature. *Analytic* averaging refers to processes in which these integrals are taken analytically; i.e., the explicit dependence of f on y is averaged out. For perturbations of a complex nature or time dependent forces, e.g. drag, tesserals, etc., investigators frequently resort to numerical quadratures of these equations.[8] The principal advantages of analytic averaging are speed and precision with respect to the averaged rates, and should be used whenever possible. For example, it is known that the average rate of the semi-major axis of a satellite due to the zonal-tesseral field of the earth is zero (in the absence of resonance), and this fact should be used in computing the total averaged rate of this element. On the other hand, numerical averaging offers high flexibility in perturbation modeling, allowing one to easily examine long-period and secular perturbations due to highly complex forces, and in varying orbital elements, e.g. Keplerian, Cartesian, equinoctial, etc. Although slightly more expensive, the authors have found such flexibility highly desirable and useful. Further, as will be seen below, numerical averaging also offers the simple development of improved averaging schemes.

Numerical solutions of the first-order system

$$\dot{\bar{x}} = \varepsilon F(\bar{x}) \qquad \dot{\bar{y}} = h(\bar{x}) + \varepsilon G(\bar{x}) \tag{7}$$

followed by the transformation $x_1 = \bar{x} + \varepsilon X(\bar{x}, \bar{y})$ and $y_1 = \bar{y} + \varepsilon Y(\bar{x}, \bar{y})$ yield first-order approximations. Error estimates of the form $|x - x_1| \sim O(\varepsilon^2 t)$ and $|y - y_1| \sim O(\varepsilon t)$ are derived in Ref. 13. It is noted that system (7) can be generally numerically integrated with stepsizes of a revolution or more and hence allows highly efficient approximate solutions, but that such solutions will secularly deviate from the actual solution over a sufficiently large interval. Further, by holding $\bar{x}$ fixed during the quadrature important long-period oscillations resulting from the interaction with high-frequency terms are lost. This situation can be improved in two ways: First one could extend the process to second-order averaging, looking for transformations of the form

$$x = \bar{x} + \varepsilon X_1(\bar{x}, \bar{y}) + \varepsilon^2 X_2(\bar{x}, \bar{y})$$

resulting in averaged equations of the form

$$\dot{\bar{x}} = \varepsilon F_1(\bar{x}) + \varepsilon^2 F_2(\bar{x}) + O(\varepsilon^3)$$

or one could improve the approximation (7) of system (3) by the examination of systems of the form

$$\dot{\bar{x}} = \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} f[x + \varepsilon X(\bar{x}, \bar{y}), \bar{y}]\, d\bar{y} \qquad \dot{\bar{y}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} h[\bar{x} + \varepsilon X(\bar{x}, \bar{y})] + \varepsilon g[\bar{x} + \varepsilon Y(\bar{x}, \bar{y}), \bar{y}]\, d\bar{y} \tag{8}$$

That system (8) yields a more precise averaged rate can be seen by noting that the goal of these averaging procedures is to determine the functions

$$\bar{x}(t) = \frac{1}{\tau}\int_{\tau-\tau/2}^{t+\tau/2} x(s)\, ds \qquad \bar{y}(t) = \frac{1}{\tau}\int_{\tau-\tau/2}^{t+\tau/2} y(s)\, ds \tag{9}$$

where τ the orbital "period" is defined by the requirement that

$$\bar{y}(t + \tau/2) - \bar{y}(t - \tau/2) = 2\pi$$

For example if y is the mean anomaly, τ is the period defined by the well-known "mean" mean motion. Differentiating Eqs. (9) with respect to t, assuming τ is independent of t, we arrive at the relations

$$\dot{\bar{x}}(t) = \frac{\varepsilon}{\tau}\int_{t-\tau/2}^{t+\tau/2} f[x(s), y(s)]\, ds \tag{10}$$

$$\dot{\bar{y}}(t) = \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} \varepsilon g[x(s), y(s)] + h[x(s)]\, ds.$$

These equations, although expressing highly precise averaged rates, are useless since knowledge of the exact solutions $x(t)$, $y(t)$ are needed to form the integrals. However, it is possible to improve over the usual approximation $x(s) = \bar{x}(t)$ inside the integral by including either first-order short-period terms available from the generating functions X and Y, or first-order secular rates

$$x(s) = \bar{x}(t) + \dot{\bar{x}}(t)(t - s)$$

It should be noted that these improvements can be easily obtained if one uses numerical averaging and extremely difficult otherwise. Such modification would yield the coupling effects mentioned earlier. From Eq. (10) we can see directly that

$$\dot{\bar{x}}(t) = \frac{\varepsilon}{\tau}\int_{-\tau/2}^{\tau/2} f[\bar{x}(t), y(s)]\, ds + O(\varepsilon^2)$$

which is precisely Eq. (3) with change of independent variable of quadrature $y \to s$.

Further, it follows from the fact that equations of the form $dx/dy = z(y)$ have solutions satisfying

$$\int_o^T x(y)\, dy = 0$$

if and only if

$$\int_o^T z(y)\, dy = 0$$

that the difference $x - \bar{x}$ will be periodic in y if and only if $\bar{x}$ satisfies Eq. (10), motivating the above discussion.

Before concluding this section, we note that functions X and Y which satisfy Eqs. (5) are not unique; in fact, to any solution of Eq. (5) one can add solutions of the homogeneous equations

$$\partial X/\partial \bar{y})h(\bar{x}) = 0 \qquad (\partial Y/\partial \bar{y})h(\bar{x}) = 0$$

and such solutions can be selected which satisfy other conditions other than zero mean, i.e.,

$$\int_o^{2\pi} X(\bar{x}, \bar{y})\, d\bar{y} = \int_o^{2\pi} Y(\bar{x}, \bar{y})\, d\bar{y} = 0$$

In fact, in analytic theory applications, such generating functions are selected so that system (3) is Hamiltonian, e.g., Brouwer, Kozai. A possible application of such generating functions which is currently being explored is the following: Assume that for some specific simplified free model such as the J_2 problem, a first order generating function S of the form (2) has been determined, e.g., Brouwer's theory. We might consider this function as the definition of an exact 1–1 transformation of dependent variables and express the full perturbation problem in this new coordinate system by the equation

$$\begin{bmatrix}\dot{\bar{x}}\\ \dot{\bar{y}}\end{bmatrix} = \frac{\partial(\bar{x},\bar{y})}{\partial(x,y)}\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} \tag{11}$$

where $\dot{x}$ and $\dot{y}$ are the full osculating derivatives. If the elimination of the first-order short-period terms due to J_2 is sufficient to achieve highly efficient numerical integration, Eq. (11) represents a very efficient method for obtaining the required averaged rates, if S is not too complicated. Further, the exact solution can be obtained by applying the transformation directly. Systems of the form (11) are currently being investigated using the Brouwer generating functions[14] and will be reported in a later paper.

A final note regarding numerical averaging is that frequently partial derivatives of the averaged elements are required with respect to certain parameters in the equations. Numerical averaging affords a simple tool to obtain such partials. For example, assume

$$\dot{x} = f(x, y, p)$$

where p is some parameter such as a ballistic coefficient. The typical variational equation

$$\frac{d}{dt}\left[\frac{\partial x}{\partial p}\right] = f_x(x,y)\frac{\partial x}{\partial p} + f_y(x,y)\frac{\partial y}{\partial p} + f_p$$

can be numerically averaged to obtain the required partial rate

$$\frac{d}{dt}\left[\frac{\partial \bar{x}}{\partial p}\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d}{dt}\left[\frac{\partial x}{\partial p}\right]dy$$

from which one can obtain the required partial derivative. This is highly useful in applications such as orbit and parameter estimation described in Sec. III.

III. Applications

The equations for the precision and averaged elements which were expressed generally in Sec. II, are in application most commonly expressed in their Lagrangian and Gaussian forms as follows.

Lagrangian Equations

$$\dot{\alpha}_i = f_i(\alpha)(\partial R/\partial a_j) \tag{12}$$

where α is the element set (a, e, i, Ω, ω, M) or some equivalent set, and R is the perturbing potential.

Gaussian Form

$$\dot{\alpha}_i = (\partial\alpha_i/\partial\dot{\bar{r}})Q + (\partial\alpha_i/\partial t) \tag{13}$$

where $\dot{\bar{r}}$ is the Cartesian velocity vector, Q is the perturbation acceleration vector in its rectangular form, and $\partial\alpha_i/\partial t$ the two-body rates of the elements.

Using the techniques of Sec. II, the method of averages can be applied to either form of the above equations, either analytically or numerically. The analytical averaging of Refs. 1–5 average the potential R in Eqs. (12) to form the averaged derivatives. The analytical averaging of Ref. 6 averages the accelerations of Eq. (13) rotated into an equinoctial coordinate frame. The numerical averaging of Ref. 8 applies a Gaussian quadrature to the right-hand side of Eqs. (12), whereas the numerical averaging of Ref. 17 applies a Gaussian quadrature to the right-hand side of Eq. (13).

One application of the averaged element rates is for rapid long-term trajectory propagation required for mission analysis and lifetime studies.[1–4,6,8] Much has been said about this application and will not be discussed here.

A second application[8–12] is the extraction of geodynamic parameters from the long-term and secular variations of mean elements. In this application, mean elements or osculating elements which are subsequently averaged are determined over many short arcs of tracking data and are "fit" with averaged dynamic equations in a least-squares process with geodynamic parameters a part of the "solve" state. The equations of condition may be expressed as

$$\bar{\alpha}_o(t_i, p+\Delta p) - \bar{\alpha}_c(t_i) = (\partial\bar{\alpha}/\partial p)\Delta p$$

where $\bar{\alpha}_o(t_i)$ are the observed mean elements from the short-arc solutions, $\bar{\alpha}_c(t_i)$ are the computed elements from the averaged equations, and p is the solve for parameters, e.g., zonal harmonics, drag coefficient, etc.

The third and most recent application of the method of averages is the estimation of element state and/or geodynamic parameters directly from tracking data.[17] The equations of condition are given by:

$$O(t_i) - O_c[t_i, \bar{\alpha}(t_i), p] = \frac{\partial O}{\partial \bar{\alpha}_o}\Delta\bar{\alpha}_o + \frac{\partial O}{\partial p}\Delta p$$

where $O(t_i)$ are the observations (range, range rate, angles, etc.), O_c are the computed observations based upon the averaged elements $\bar{\alpha}(t_i)$, and the solve parameters are initial mean elements $\bar{\alpha}_o$ and physical parameters p.

Note that the observations are in osculating element space whereas the computed values are in averaged element space. Thus, the least-squares estimation process must absorb the short periodic variations between the observed and computed values. The standard error of fit of this process is thus much larger than standard osculating element estimators; however, this process is quite efficient and yields accurate mean elements (see Sec. V).

IV. Use of Mean Elements in the Method of Averages

Careful considerations must be given to the initial conditions that are required for the integration of the averaged differential equations developed in Sec. II. If the mean state of the orbit can be determined directly from mission analysis considerations or from the orbit determination process (discussed in Secs. II and V), then the averaged equations may be integrated for prediction purposes using this mean state as initial conditions. Alternative procedures are as follows:

1) Using a technique similar to that suggested by Uphoff in Ref. 8, define

$$\bar{\alpha}(t) = \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2}\alpha(s)\,ds \tag{14}$$

This function can be shown to satisfy the averaged differential equation of Sec. I

$$\dot{\bar{\alpha}}(t) = (1/\tau)\int_{t-\tau/2}^{t+\tau/2}\alpha(s)\,ds \tag{15}$$

if it is assumed τ is independent of t.

From this definition, it follows immediately that

$$\bar{\alpha}_o = \frac{1}{\tau}\int_{t_o-\tau/2}^{t_o+\tau/2}\alpha(s)\,ds \tag{16}$$

Difficulties arise in the definition of the quadrature interval τ. First one must note that the interval τ of Eq. (14) must be consistent with the interval τ of Eq. (15). Since τ is defined as a function of $\bar{\alpha}$, the procedure is to solve this quadrature by iteration, using a numerically integrated $\alpha(t)$ defined by the osculating dynamics; i.e., using an initial estimate of τ, a set of elements $\bar{\alpha}_o$ could be found defining a new τ, etc.

2) Let α_o be the initial conditions corresponding to the osculating dynamical system

$$\dot{\alpha} = f(t, \alpha) \tag{17}$$

i.e., assume a set of osculating initial conditions are known. Then corresponding initial conditions for (15) can be determined by a technique known as "precise conversion of elements" (PCE).[17]

Integrate Eq. (17) with initial conditions defined by α_o over the period τ. Let $\bar{\alpha}_{o1}$ be a first guess at $\bar{\alpha}_o$ that satisfies Eq. (15). Then a corrected set $\bar{\alpha}_o = \bar{\alpha}_{o1} + \Delta\bar{\alpha}_o$ can be computed in the least-squares sense from

$$\Delta\bar{\alpha}_o = (A_i^T A_i)^{-1} A_i^T \Delta\alpha_i$$

where subscript i indicates summation over i, and

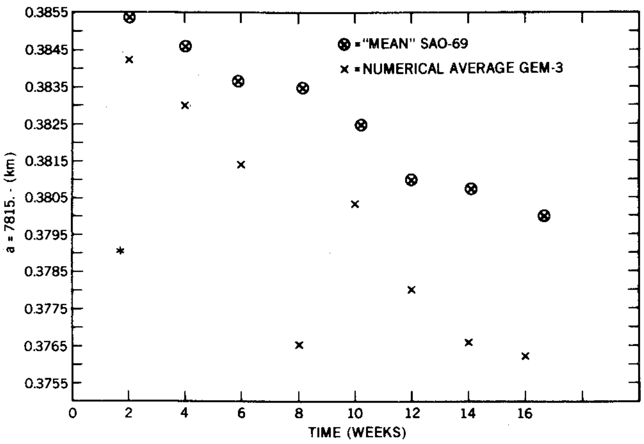

Fig. 1 ESSA-8 mean semi-major axis.

$$\Delta a_i = \alpha(t_i, \alpha_o) - \bar{\alpha}(t_i, \bar{\alpha}_{o1})$$

$$A_i = \frac{\partial \alpha_i}{\partial \bar{\alpha}_o}, \quad i = 1, \ldots, m \text{ over } \left[\frac{-\tau}{2} > \frac{\tau}{2}\right]$$

The resultant set of elements $\bar{\alpha}_o$ is a set of elements that satisfies the differential equations (15) and that minimizes the squares of the differences between the element set α and $\bar{\alpha}$.

3) Another approach to finding "mean" elements to satisfy (15) when a set of osculating elements that satisfy Eq. (17) is known was given by Douglas et al.[11] This method utilizes Kaula's formulation[16] to remove to first order the short periodic and m-daily effects, and then takes the arithmetic mean of the resultant elements, yielding second-order accuracy. In utilizing these elements with Eq. (15) a multiple quadrature would be required such that both short periodic and daily effects are averaged in the equations of motion.

V. Numerical Experience and Results

A. Accuracy and Efficiency Comparisons

The following is a summary of accuracy and efficiency comparisons on the combined orbit estimation and orbit prediction problem using averaged equations of motion, Cowell equations, and the Brouwer general perturbation equations.

A series of experiments were made to examine the relative accuracy and efficiency of the preceding three theories using the ESSA–8 satellite orbit which has orbital characteristics of 1400 km altitude, near circular, and an inclination of 102°.

The experiments were carried out as follows. 1) The orbit was estimated from Minitrack data over 2–4 day arcs with each of the three theories. 2) Each resultant orbit was then predicted for a two-week period using a compatible equation for orbit prediction and orbit estimation. 3) Using the predicted elements from step two as the initial conditions, the orbit was again estimated from Minitrack data utilizing data at the end of the two week interval. 4) Along-track errors were computed from the final corrections to the state vector.

These steps were carried out with several arcs for each of the theories, and the results are tabulated in Table 1. The tabulated results are averages of all of the data arcs for each set of equations. Because of the stability of this orbit, all methods are shown to predict with accuracies of less than 1 sec whereas the required accuracy is 10 sec over the two-week period. The efficiency estimates in terms of computer time are shown for both the orbit determination and orbit prediction runs. As expected, the Cowell equations took the most computer time, and it is anticipated that due to the density of data over the data arc, the Brouwer equations took more time than the averaged equations during the orbit estimation process. The use of interpolation with the Brouwer approach might have reduced the computer time for this case somewhat. These experiments will be continued on other satellites having orbits requiring more complex perturbation models.

Table 1 ESSA-8 prediction errors[a]

Method	Along track error[b] (sec)	Computer time (Sec) Orbit determination	Orbit pred.
Cowell	0.23	74	25
Averaged dynamics	0.10	21	1
Brouwer	0.08	35	1

[a] Orbit characteristics: 1400 km circular, 102 deg inclination; Data arc: 2–4 days; Prediction period: 2 weeks; Main perturbations: J_2, J_3, J_4.
[b] Required accuracy: 10 sec.

B. Tracking Data Determination of Mean Elements

Mean orbital elements for the ESSA–8 satellite were determined by the following two procedures. The first approach used to obtain accurate "mean" elements was to reduce short arcs (approximately two days) of tracking data (minitrack) to osculating elements using a high precision integration of the Cowell equations of motion, and then to convert the osculating elements to mean elements by the approach described in Sec. IV-3, i.e., making use of the Kaula formulation for the short period and m-daily effects. These elements are referred to as MEANEL elements, so called because of the use of the computer program of the same name which incorporates this procedure. This technique is a costly operation on the computer because of the high-frequency nature of the perturbations in the osculating orbit.

The second technique to obtain accurate "average" elements fits the averaged equations of motion directly to the tracking data and lets the least-squares process in the data reduction algorithm average out the short-period terms. Figure 1 shows the comparison for the semi-major axis over a 16-week period. It's first noted that both techniques show a spread of 10 meters or less and removing the secular trend gives a variance of 1 meter. The translation of several meters can be attributed to several differences. 1) No drag or resonance perturbations where modeled in the fitting of averaged dynamics to tracking data (this was an oversight and not a limitation of the averaged dynamic approach). 2) Different values of GM were used in the two approaches. 3) m-daily effects not being removed in the numerical averaging technique. Figure 2 shows the corresponding comparisons for the eccentricity.

C. Averaging Interval and Rotating Primary

The averaging technique may be considered to be a high-frequency filter which filters out all perturbations with frequencies less than and commensurate with one period. One of the highlighting features of the numerical averaging technique as opposed to analytical averaging is the ability of the numerical averaging to easily accommodate a general perturbation model including tesseral harmonics, third-body perturbations, drag, solar radiation pressure, etc. The question thus arises as to how to design an optimal filter in the presence of all

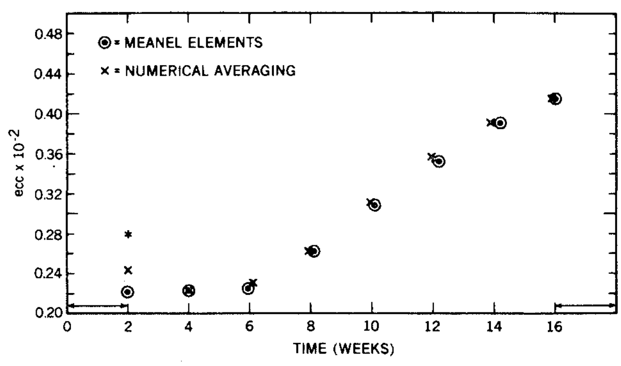

Fig. 2 ESSA-8 mean eccentricity.

perturbations, particularly those that have a time dependence that is small compared to the orbital period. This problem was addressed by Lorell and Liu in Ref. 5 in discussing the effect of the rotation of the primary. In their Lunar Orbiter analysis, the rotation of the primary (28 days for the Moon) was large compared to the orbital period of Lunar Orbiter. For the multitude of Earth orbiting satellites with period of 2 hours or less, the problem is rather severe.

When the perturbation model includes the zonals only, since the rotation of the primary does not appear in the formulation, an averaging interval that is an even multiple of the "mean" orbital period is satisfactory. However, when the tesserals are included in the perturbation model, this is no longer the case.

Since much attention has been given to the ability of the averaging technique to minimize along track errors, let us consider the effect of the rotating primary on the integration of the semi-major axis.

The Lagrangian equation for semi-major axis is:

$$\dot{a} = (2/na)(\partial R/\partial M)$$

Using Kaula's expansion for the potential we have:

$$V_{lm} = [\mu a_e^{l}/(a^{l+1})]F_{lmp}(i)G_{lpq}(e)S_{lmpq}(\omega, M, \Omega, \theta)$$

where l and m are, respectively, the degree and order of the harmonics,

$$S_{lmpq} = \begin{bmatrix} C_{lm} \\ -S_{lm} \end{bmatrix}_{l-m\,\mathrm{odd}}^{l-m\,\mathrm{even}} \cos[(l-2p)\omega + (l-2p+q)M + m(\Omega-\theta)] + \begin{bmatrix} S_{lm} \\ C_{lm} \end{bmatrix}_{l-m\,\mathrm{odd}}^{l-m\,\mathrm{even}} \sin[(l-2p)\omega + (l-2p+q)M + m(\Omega-\theta)]$$

Here, $\theta = \theta_o + \dot{\theta}t$, i.e., the rotation of the primary is represented by $\dot{\theta}$. Obviously, the zonals ($m = 0$) are independent of this rotation. Let us simplify the analysis to considering only C_{2200} with $S_{22} = 0$. Then

$$V_{22} = \frac{\mu a_e^2}{a^3} F(i)G(e)C_{22}[\cos 2x + \sin 2x]$$

where

$$x = \omega + M + \Omega - \theta$$

and

$$\frac{\partial V_{22}}{\partial M} = \frac{2\mu a_e^2}{a^3} F(i)G(e)C_{22}[\cos 2x - \sin 2x]$$

hence

$$\dot{a} = \frac{4\mu a_e^2}{na^4} F(i)G(e)C_{22}[\cos 2x - \sin 2x]$$

letting

$$M = M_o + nt$$

$$b = 2(n - \dot{\theta})$$

we obtain the final expressions

$$\dot{a} = \frac{4\mu a_e^2}{na^4} F(i)G(e)C_{22}[\cos bt - \sin bt]$$

$$\dot{\bar{a}} = \frac{\mu a_e^2}{\pi a^4(n-\dot{\theta})} F(i)G(e)C_{22}[\cos bt - \sin bt]$$

Thus, the effect of averaging the Lagrangian equation for a is merely to alter the amplitude of the derivative while the frequency remains unchanged. Evaluating the amplitudes with

$$F(i) = \tfrac{3}{4}(1+\cos i)^2$$

and

$$G(e) = 1 - (5e^2/2)$$

shows that the amplitude of the averaged derivative is reduced by a factor of approximately 16.

This phenomenon is demonstrated numerically in Fig. 3 where the averaged equations with C_{22} in the force model were numerically integrated with an unrealistically small step size ($h = 1$ min). The twice per revolution frequency is clearly seen. However, when only the zonals were included in the force model, clearly $\bar{a}$ is constant since $\dot{\bar{a}} = 0$.

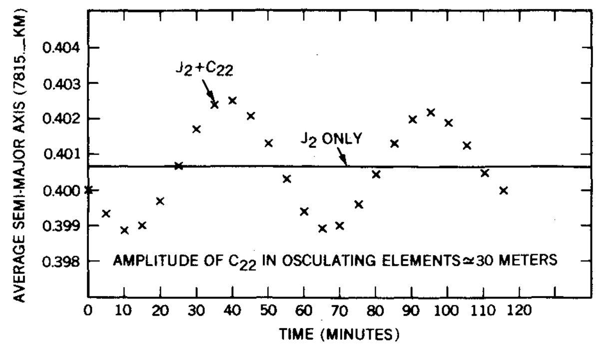

Fig. 3 ESSA-8 average semi-major axis, numerical integration of average equations.

Thus, what does one do with numerical averaging when tesseral (m-daily or resonant) effects are desired in the integration of $\dot{\bar{a}}$? One approach would be to perform multiple quadratures over the main and secondary frequencies due to the rotating primaries (i.e., over n and $n-\dot{\theta}$). An easier approach is to make use of knowledge from the analytical expressions, and set the derivatives due to particular terms in the potential identically to zero.

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